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TECHNICAL TRANSLATION

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AN APPROXIMATION METHOD FOR THE INTEGRATION OF THE EQUATIONS OF A NONSTATIONARY LAMINAR BOUNDARY LAYER IN AN INCOMPRESSIBLE FLUID

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AN APPROXIMATION METHOD FOR THE INTEGRATION OF THE EQUATIONS OF A NONSTATIONARY LAMINAR BOUNDARY LAYER IN AN INCOMPRESSIBLE FLUID*

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For the solution of the equations of a nonstationary laminar boundary layer, the method of successive approximations /1/ is used. However, because of the considerable computational difficulties, such a method permits the examination of only a limited number of problems. Therefore, attempts have recently been made to find approximate solutions for problems of a non-stationary boundary layer in a simpler manner. Thus, E. M. Dobryshman /2/, using the concept of a "layer of finite thickness" and basing himself on the method proposed by M.E. Shoets /3/, discussed a number of cases of continual movement of a fluid in a boundary layer. Analogous situations, but of a somewhat different form, became the basis of the computation of a nonstationary boundary layer proposed by S.M. Targ /4/.

This article gives the approximate solution of the equations of a non-stationary laminar boundary layer in an incompressible fluid, based on the utilization of the equations of momenta and a certain one-parameter family of velocity profiles. As is well known, this method of approximating the solution yields good results in the study of stationary problems.

1. Formulation of the Problem: The equations of a plane nonstationary laminar boundary layer in the absence of volume forces has the form

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = U\frac{\partial U}{\partial x} + \frac{\partial U}{\partial t} + \nu\frac{\partial^2 u}{\partial y^2}, \qquad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (1.1)$$

where $u(x, y, t)$ and $v(x, y, t)$ are the longitudinal and transverse velocity projections in the sections of the boundary layer on the coordinate axes x and y, directed along the tangent and normal to the streamlined surface, t is time, $U(x, t)$ the given longitudinal velocity on the outer boundary, and $\nu = \mu/\rho$ the kinematic coefficient of viscosity. The unknown functions u and v must satisfy the boundary conditions

$$u = v = 0 \text{ for } y = 0, \qquad u \to U \text{ for } y \to \infty \qquad (1.2)$$

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*Translated from Prikladnaia Matematika i Mekhanika, Institute of Mechanics of the Academy of Sciences of the USSR, vol. 21, 1957, pp. 615-623.

and in addition the distribution of velocities at the initial instant of time must be given.

It is easy to obtain /5/ the main integral relation of system (1.1), known as the equation of momentum;

$$\frac{\partial \delta^{**}}{\partial x} + \frac{1}{U}\frac{\partial \delta^{*}}{\partial t} + \frac{1}{U}\frac{\partial U}{\partial x}(2\delta^{**} + \delta^{*}) + \frac{1}{U^2}\frac{\partial U}{\partial t}\delta^{*} = \frac{\nu}{U^2}\left(\frac{\partial u}{\partial y}\right)_{y=0} \qquad (1.3)$$

Here δ^* is the thickness of displacement and δ^{**} the thickness of the loss of momentum.

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$$\delta^* = \int_0^\infty \left(1 - \frac{u}{U}\right) dy, \qquad \delta^{**} = \int_0^\infty \frac{u}{U}\left(1 - \frac{u}{U}\right) dy \qquad (1.4)$$

Pursuing the idea of the computation of a boundary layer by means of the equations of momenta /5/, we will assume that velocity profiles in a boundary layer can be represented in the form of some one parameter family of velocities

$$\frac{u}{U} = \Phi'(\eta, \beta), \qquad \eta = \frac{yA(\beta)}{\delta^*} \qquad (1.5)$$

where the dash denotes differentiation with respect to η, and β a parameter depending on x and t. From the first formula (1.4) it follows that

$$A(\beta) = \int_0^\infty [1 - \Phi'(\eta, \beta)]\, d\eta \qquad (1.6)$$

Keeping in mind the approximate solution of system (1.1), we shall demand that the velocities profile (1.5) satisfy, instead of (1.1), the integral impact relation (1.3) and the first equation (1.1), when $y = 0$. This can be achieved by a proper choice of the magnitudes δ^* and β in (1.5). Thus, letting in the first equation (1.1) $y = 0$, and substituting for the longitudinal velocity component u its value according to formula (1.5), we shall get

$$-\nu\left(\frac{\partial^2 u}{\partial y^2}\right)_{y=0} = U\frac{\partial U}{\partial x} + \frac{\partial U}{\partial t} = -\nu\frac{U}{\delta^{*2}}A^2(\beta)\Phi'''(0, \beta) \qquad (1.7)$$

We introduce the notation

$$f = -A^2(\beta)\Phi'''(0, \beta) = \frac{1}{U}\left(U\frac{\partial U}{\partial x} + \frac{\partial U}{\partial t}\right)\frac{\delta^{*2}}{\nu} \qquad (1.8)$$

The dimensionless quantity f, which is a known function of β, will be taken for the parameter which characterizes the deformation of the chosen velocity profile. It is evident that when $\partial U/\partial t = 0$ the magnitude f becomes

the known parameter of a stationary boundary layer /5/, if in the latter we substitute for δ^{**}, δ^{*}.

We shall now satisfy the momentum equation (1.3). To this end we shall express the magnitudes δ^{**} and $(\partial u/\partial y)_{y=0}$ which are comprised in (1.3) in terms of f and δ^{*}.

Taking into account (1.4), we shall have

$$\delta^{**} = \frac{\delta^{*}}{A(\beta)} \int_0^\infty \Phi'(\eta, \beta)[1 - \Phi'(\eta, \beta)]\, d\eta = \delta^{*} \frac{B(\beta)}{A(\beta)} \tag{1.9}$$

where

$$B(\beta) = \int_0^\infty \Phi'(\eta, \beta)[1 - \Phi'(\eta, \beta)]\, d\eta \tag{1.10}$$

Since β and f are related by formula (1.8), we can let

$$\frac{\delta^{**}}{\delta^{*}} = \frac{B(\beta)}{A(\beta)} = h(f) \tag{1.11}$$

Further

$$\left(\frac{\partial u}{\partial y}\right)_{y=0} = \frac{UA(\beta)\Phi''(0, \beta)}{\delta^{*}} = \frac{U\zeta(f)}{\delta^{*}} \tag{1.12}$$

where $\zeta(f) = A(\beta)\Phi''(0.\beta)$. We shall substitute in the momentum equation (1.3) the magnitudes δ^{**}, $(\partial u/\partial y)_{y=0}$, expressed in terms of δ^{*}f, with the aid of formulas (1.11) (1.12). We shall then get

$$\frac{\partial h(f)\delta^{*}}{\partial x} + \frac{1}{U}\frac{\partial \delta^{*}}{\partial t} + \frac{1}{U}\frac{\partial U}{\partial x}\delta^{*}[2h(f) + 1] + \frac{1}{U^2}\frac{\partial U}{\partial t}\delta^{*} = \frac{\nu\zeta(f)}{U\delta^{*}} \tag{1.13}$$

In this way the distribution of velocities (1.5) will satisfy the aforementioned conditions, provided the parameter f and the thickness of displacement δ^{*} satisfy equations (1.8), (1.13).

2. <u>The Choice of a One Parameter Family of Velocity Profiles in the Sections of a Boundary Layer, Approximate Computation of a Nonstationary Laminar Boundary Layer.</u> Analogously with the well known methods for a stationary layer /5, 6/, we shall propose for the approximate computation of a nonstationary boundary layer two families of velocity profiles.

1. Let us use the exact solution of the equations of a boundary layer which corresponds to the distribution of velocities on the outer boundary of the layer, in the form of a power monomial $U = x^m$. Then in formula (1.5) Φ will be the well known function of Hartree /7/, satisfying the equation

$$\Phi''' + \Phi\Phi'' = \beta(\Phi'^2 - 1), \qquad \beta = \frac{2m}{m+1} \tag{2.1}$$

and the boundary conditions $\Phi(0) = \Phi'(0) = 0,\ \Phi'(\infty) = 1.$. In the left part of Table 1 the values of $\beta, f, A(\beta), h(f), \zeta(f)$ are given for the whole range of variations of m, in particular from the value m = -0.0904 (β = - 0.1988), corresponding to the break, up to m = ∞ (β = 2.00).

On Figure 1 the relations h(f) and ζ(f) are represented, which turn out to be close to the linear $h(f)$. Thus, since the function h(f) varies negligibly, an assumption can be made that is sometimes used in the theory of a boundary layer, namely, that the velocity profile deformation has no bearing on the relation of such integral magnitudes as δ^{**} and δ^{*}. Thus we shall consider h(f) = α_1, as a constant magnitude whose choice is somewhat arbitrary. In further work, we shall take α_1, equal to the mean value h(f) = 0.356. However, it should be noted that for the determination of the break of a boundary layer, more precise results are given by α_1, equal to the break value h(f) = 0.248. Passing to the relation $\zeta(f)$, we shall represent it in the form

$$\zeta(f) = \alpha_2 + \alpha_3 f + \varepsilon(f) \qquad (2.2)$$

where α_2, α_3 are constants, and ε (f) a small correction. If we let $\alpha_2 = 0.573$, $\alpha_3 = 0.52$ (the unbroken straight line in Fig. 1), then in the interval of the values of f which we need for our purpose. the inequality $|\varepsilon(f)| < 0.0125\,\alpha_2$ will hold. This inequality permits us to consider $\zeta(f)$ as a linear function. Beside the coefficient values indicated in the majority of cases it is convenient to take $\alpha_2 = 0.56$, $\alpha_3 = 0.5$ (the unbroken straight line in Fig. 1). This yields $|\varepsilon(f)| < 0.058\,\alpha_2$. .

2. Let us use the first approximation (for small t's) to the exact solution of the problem of the growth of a boundary layer, having a velocity on the boundary, given in the form U = t^m W(x). In this case the equation of a first approximation, with small t, has the form

$$\frac{\partial u}{\partial t} = \frac{\partial U}{\partial t} + \nu \frac{\partial^2 u}{\partial y^2} \qquad (2.3)$$

and the boundary and initial conditions will be u = 0 for y = 0, u $\to$ U for y $\to \infty$, u = U for t = 0, y $\neq$ 0.

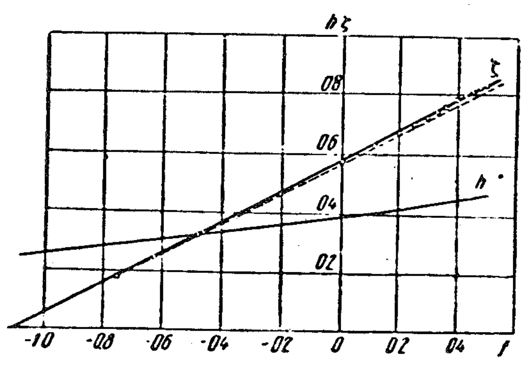

Fig. 1

The solution of the formulated problem was obtained /8/ in the form (1.5), where the function Φ was determined from the equation

$$\Phi''' + 2\eta\Phi'' - \frac{4\beta}{2-\beta}(\Phi' - 1) = 0, \qquad \beta = \frac{2m}{m+1} \tag{2.4}$$

and satisfied the conditions.

TABLE 1.

β	f	$A(\beta)$	$h(f)$	$\zeta(f)$	β	f	$A(\beta)$	$h(f)$	$\zeta(f)$
−0.1988	−1.106	2.359	0.2479	0.000	−2.000	−1.571	0.884	0.293	0.000
−0.19	−0.7653	2.007	0.2874	0.1726	−1.640	−1.240	0.826	0.317	0.138
−0.18	−0.6302	1.871	0.3035	0.2404	−1.335	−0.980	0.780	0.336	0.245
−0.16	−0.4667	1.708	0.3231	0.3254	−1.078	−0.771	0.740	0.352	0.330
−0.14	−0.3570	1.597	0.3375	0.3825	−0.857	−0.600	0.705	0.365	0.400
−0.10	−0.2085	1.444	0.3566	0.4608	−0.667	−0.457	0.675	0.377	0.457
0.00	0.0000	1.217	0.3861	0.5715	−0.500	−0.337	0.646	0.386	0.505
0.10	0.1166	1.080	0.4027	0.6340	−0.222	−0.145	0.600	0.402	0.580
0.20	0.1936	0,984	0.4146	0.6759	0.000	0.000	0.560	0.414	0.637
0.30	0.2490	0.911	0.4237	0.7058	0.400	0.243	0.490	0.435	0.729
0.40	0.2910	0.853	0.4302	0.7286	0.667	0.393	0.442	0.448	0.785
0.50	0.3232	0.804	0.4353	0.7459	0.857	0.494	0.405	0.456	0.822
0.60	0.3502	0.764	0.4397	0.7609	1.000	0.566	0.376	0.463	0.849
0.80	0.3909	0.699	0.4463	0.7829	1.200	0.663	0.332	0.471	0.884
1.00	0.4199	0.648	0.4506	0.7987	1.331	0.724	0.300	0.476	0.905
1.20	0.4421	0.607	0.4546	0.8109	1.430	0.767	0.276	0.480	0.920
1.60	0.4734	0.544	0.4595	0.8274	1.500	0.798	0.257	0.483	0.931
2.00	0.4960	0.498	0.4638	0.8401	1.555	0.822	0.242	0.485	0.940
					2.000	1.000	0.000	0.500	1.000

In the right part of Table 1, the values $\beta, f, A(\beta), h(f), \zeta(f)$ were given for the indicated distribution of velocities when f varies between the limits -1.571 and 1.0, which corresponds to changes of m from -0.5 (break profile) to ∞. The graphs of the functions h(f) and $\zeta(f)$, are represented in Fig. 2. First it follows from Fig. 2 that the dependencies h(f) and ζ(f) behave in the same way as in the case of Hartree's profile. Thus, the function h(f) varies negligibly and we can take h(f) = α_1 = 0.397 (mean value). The function $\zeta(f)$ can, as in the aforementioned case, be considered linear and we can let $\alpha_2 = 0.620, \alpha_3 = 0.401$. With this the inequality $|\epsilon(f)| < 0.03\alpha_2$ will be satisfied.

Now that the velocity profiles have been chosen in the boundary layer and the dependencies h(f), ζ(f) are known, we shall transform (1.8) and (1.13) into a single differential equation relative to δ^*. To this end we shall substitute in (1.13), instead of h(f), the constant α_1, and taking into consideration relations (1.8), (2.2) when $\varepsilon(f) = 0$, we shall express the right member of (1.13) in terms of δ^*. We shall then get

$$\frac{\partial \varphi}{\partial t} + aU\frac{\partial \varphi}{\partial x} + \left(b\frac{\partial U}{\partial x} + c\frac{1}{U}\frac{\partial U}{\partial t}\right)\varphi = p \qquad \left(\varphi = \frac{\delta^{*2}}{\nu}\right) \tag{2.5}$$

where

$$a = \alpha_1, \qquad b = 2(2\alpha_1 + 1 - \alpha_3), \qquad c = 2(1 - \alpha_3), \qquad p = 2\alpha_2 \tag{2.6}$$

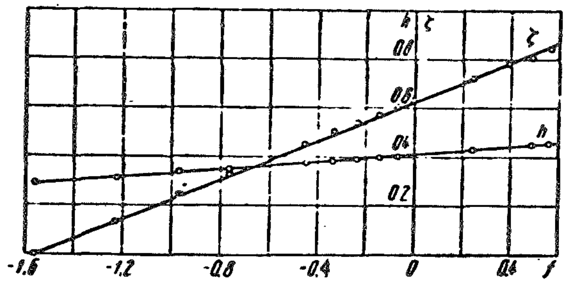

Fig. 2

For the single-valued determination of φ an additional condition must be given, which in its general form reduces to the requirement that the integral surface of equation (2.5) intersect the given curve in the space (x, t, φ), where the given curve is not a characteristic.

The determination of φ by means of the integration of equation (2.5) under the corresponding conditions fully solves the problem of the approximate computation of a nonstationary boundary layer. In fact, when φ and consequently δ^* are known, the parameter f is easily determined with the aid of formula (1.8). Further, using Table 1, we can find the values of β, $A(\beta)$, and determine according to formula (1.5) the longitudinal velocity component u in the boundary layer. The transverse velocity component v is found from the second equation of system (1.1). The friction stress on the streamlined surface τ_0 is expressed in terms of φ, with the aid of formula (1.12).

$$\tau_0 = \mu\left(\frac{\partial u}{\partial y}\right)_{y=0} = \sqrt{\mu\rho}U\left[\frac{\alpha_2}{\sqrt{\varphi}} + \alpha_3\left(\frac{\partial U}{\partial x} + \frac{1}{U}\frac{\partial U}{\partial t}\right)\sqrt{\varphi}\right] \tag{2.7}$$

and the break condition of the boundary layer has the form

$$\tau_0 = 0 \quad \text{or} \quad \frac{1}{U}\left(U\frac{\partial U}{\partial x} + \frac{\partial U}{\partial t}\right)\varphi = -\frac{\alpha_2}{\alpha_3} \tag{2.8}$$

In the above derivation of equation (2.5), the assumptions $h(f) = \alpha_1 = \text{const}$ and $\epsilon(f) = 0$ have been made. Greater exactness can be achieved in the proposed computation by means of successive approximations, taking for the first approximation equations (2.5). Then in the

second approximation, $h(f)$ and $\epsilon(f)$ must be considered as given by the first approximation, etc. In the result, it would be necessary to integrate equations of type 5 every time.*

3. Examples. 1°. Let us assume that a round cylinder starts moving from rest with a velocity $U_0 t^n$. Under unbroken streamlining we shall have, as known, on the boundary of the boundary layer $U = 2U_0 t^n \sin\frac{x}{r}$, where r is the radius of the cylinder. We determine the path s_0, which will pass through the cylinder from the initial instant of movement up to the formation of a break in the rear critical point. To this end we shall substitute the value of the velocity U and that of its derivatives when $x = \pi r$ in the equation (2.5). We shall then get

$$\frac{d\varphi}{dt} + \left(c\frac{n}{t} - 2b\frac{U_0}{r}t^n\right)\varphi = p \tag{3.1}$$

Integrating equation (3.1) when $c = 1$ and taking into consideration that $\varphi = 0$ at the initial instant of time $t = 0$, we shall get the law of increase of δ^* at the critical rear point of the cylinder

$$\varphi = \frac{p}{2b}\frac{r}{U_0}t^{-n}\left[\exp\left(2b\frac{U_0}{r}\frac{t^{n+1}}{n+1}\right)-1\right] \tag{3.2}$$

Using condition (2.8) and the expression (3.2), we shall get an equation determining the beginning of the break of the boundary layer.

$$\left[\gamma\left(2\frac{U_0}{r}\frac{t^{n+1}}{n+1}\right)^{-1}-1\right]\left[\exp\left(2b\frac{U_0}{r}\frac{t^{n+1}}{n+1}\right)-1\right] = -\frac{a_2}{a_3}\frac{b}{p}, \qquad \gamma = \frac{n}{n+1} \tag{3.3}$$

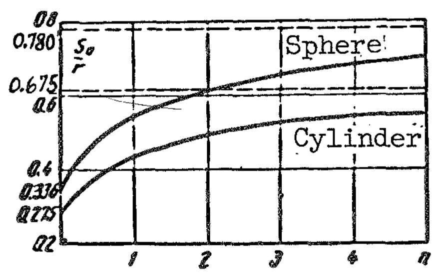

Fig. 3

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*It should be noted that the equations for the thickness of a boundary layer $\delta(x,t)$, obtained by E.M. Dobryshman /2/ and S.M. Targ /4/ can be reduced to the equation (2.5), if we take approximately $\delta = 3\delta^*$. Thus, in various cases the coefficients a, b, c, p, assume the following values: 1) Hartree's profile (the unbroken straight line in Fig. 1 for $\alpha_1 = 0.356$) $a = 0.356$, $b = 2.382$, $c = 0.960$, $p = 1.150$; 2) Hartree's profile (broken straight line in Fig. 1, for $\alpha_1 = 0.356$) $a = 0.356$, $b = 2.422$, $c = 1.000$, $p = 1.120$; 3) the profile of velocities determined by equation (2.4), $a = 0.397$, $b = 2.784$, $c = 1.198$, $p = 1240$; 4) E.M. Dobryshman, $a = 0.375$, $b = 2.250$, $c = 1.000$, $p = 0.667$; 5) S. M. Targ, $a = 0.429$, $b = 2.398$, $c = 1.000$, $p = 1.100$.

In the case n = 0, formulas (3.2) and (3.3) hold for any c. This enables us to compare, when n = 0, the values of s_0, which were obtained by various computational methods. If we take α, equal to the break value h(f), then using Hartree's formula (the broken line in Fig. 1), we shall get $s_0/r = 0.275$. Almost the same result is obtained using the straight unbroken line in Fig. 1. A calculation based on the relations in Fig. 2 gives the breaks a little later, e.g. $s_0/r = 0.290$. At the same time S.M. Targ /4/ obtained $s_0/r = 0.25$. All the indicated values s_0/r turn out to be somewhat smaller than $s_0/r = 0.320$, which is obtained by an immediate integration of the equations of a nonstationary boundary layer /1/. Thus, in the given case, the momentum method, based on the distribution of velocities obtained from 2.4), yields results which are closest to the exact result. In Fig. 3 the relation $\frac{s_0}{r}$ depending on n, is represented. It has been computed with the aid of the expression (3.3). Here the coefficients in (3.3) correspond to Hartree's function (the broken line in Fig. 1). It follows from Fig. 3 that as the exponent n increases, the value $\frac{s_0}{r}$ increases quickly for small n's, and gradually ceases to vary for large n's.

2^0. We consider the classes of problems for which (2.5) is transformed into an ordinary differential equation. We shall introduce the magnitudes U_0, , τ , λ having the dimensions

$$[U_0] = \frac{L}{T}, \qquad [\tau] = \frac{1}{T}, \qquad [\lambda] = \frac{1}{L} \tag{3.4}$$

where L and T are the scales adopted for length and time.

Then one can get the value of φ in the following cases:

$$1) \qquad U = U_0 k(\xi)\, t^m \tau^m, \qquad \xi = x t^{-(m+1)} \frac{\tau^{-m}}{U_0} \tag{3.5}$$

$$\varphi = \frac{t}{[ak - (m+1)\xi]} \exp\left[-\int \frac{(m+2) + cm + (b-a)k' - c(m+1)\xi k'/k}{ak - (m+1)\xi} d\xi\right] \times$$
$$\times \left\{D + p\int \exp\left[\int \frac{(m+2) + cm + (b-a)k' - c(m+1)\xi k' k}{ak - (m+1)\xi} d\xi\right] d\xi\right\} \tag{3.6}$$

In formula (3.6) D is the constant of integration.

$$2) \qquad U = U_0 k(\xi)\, e^{\tau t}, \qquad \xi = x e^{-\tau t} \frac{\tau}{U_0} \tag{3.7}$$

$$\varphi = \frac{1}{\tau} \frac{1}{ak - \xi} \exp\left[-\int \frac{1 + c + (b-a)k' - c\xi k'/k}{ak - \xi} d\xi\right] \times$$
$$\times \left\{D + p\int \exp\left[\int \frac{1 + c + (b-a)k' - c\xi k'/k}{ak - \xi} d\xi\right] d\xi\right\} \tag{3.8}$$

$$3) \qquad U = U_0 k(\xi)\, x^m \lambda^m, \qquad \xi = t x^{m-1} U_0 \lambda^m \tag{3.9}$$

$$\varphi = \frac{x^{1-m}\lambda^{-m}}{U_0[1+a(m-1)\xi k]}\exp\left[-\int\frac{2a(1-m)k+bmk+(b-a)(m-1)\xi k'+ck'/k}{1+a(m-1)\xi k}\,d\xi\right]\times$$
$$\times\left\{D+p\int\exp\left[\int\frac{2a(1-m)k+bmk+(b-a)(m-1)\xi k'+ck'/k}{1+a(m-1)\xi k}\,d\xi\right]d\xi\right\} \qquad (3.10)$$

4) $$U = U_0 k(\xi)e^{\lambda x}, \qquad \xi = tU_0\lambda e^{\lambda x} \qquad (3.11)$$

$$\varphi = \frac{e^{-\lambda x}}{\lambda U_0(1+a\xi k)}\exp\left[-\int\frac{(b-2a)k+(b-a)\xi k'+ck'/k}{1+a\xi k}\,d\xi\right]\times$$
$$\times\left\{D+p\int\exp\left[\int\frac{(b-2a)k+(b-a)\xi k'+ck'/k}{1+a\xi k}\,d\xi\right]d\xi\right\} \qquad (3.12)$$

3°. Let us assume that at the initial instant of time $t = 0$ a semi-infinite plate starts moving from rest on the boundary of a boundary layer with the velocity $U = U_0 + \sigma(1 - \cos\omega t)$. Together with this we shall take into account that the main velocity U_0 is subject to small fluctuations ($U_0 \gg \sigma$). Then, if for the convenience of the solution of the problem we introduce on the boundary of the layer the complex velocity $\mathbf{U} = U_0 + \sigma(1 - e^{i\omega t})$ (here and from here on complex magnitudes will be denoted in semithick print), then we can let $\mathbf{U} \approx U_0$ and $\frac{d\mathbf{U}}{dt} = i\sigma e^{i\omega t}$. Substituting these expressions in equation (2.5) and considering the magnitude φ as a complex magnitude, we will have

$$\frac{\partial\varphi}{\partial t} + aU_0\frac{\partial\varphi}{\partial x} - c\,\frac{i\sigma\omega}{U_0}\,e^{i\omega t}\varphi = p \qquad (3.13)$$

Passing in equation (3.13) to the parameter f according to formula

$$f = \frac{1}{U}\left(U\frac{\partial U}{\partial x} + \frac{\partial U}{\partial t}\right)\varphi = -\frac{i\sigma\omega}{U_0}\,e^{i\omega t}\varphi \qquad (3.14)$$

we shall come to the following equation relative to f:

$$\frac{\partial f}{\partial t} + aU_0\frac{\partial f}{\partial x} - \left(i\omega + c\,\frac{i\sigma\omega}{U_0}\,e^{i\omega t}\right)f = -p\,\frac{i\sigma\omega}{U_0}\,e^{i\omega t} \qquad (3.15)$$

Finally, disregarding in (3.15) the second term in the parenthesis, we shall have

$$\frac{\partial f}{\partial t} + aU_0\frac{\partial f}{\partial x} = i\omega f - p\,\frac{i\sigma\omega}{U_0}\,e^{i\omega t} \qquad (3.16)$$

Equation (3.16) must be integrated with the condition that f vanishes both at the initial instant of time $t = 0$, as well as at the front edge of the plate when $x = 0$. To this end we shall apply the t operator method. Then after the usual calculations we have the following expression for the parameter $f(x, t)$, equal to Re $f(x, t)$:

$$f(x,t) = \frac{p}{a}\,\frac{\sigma\omega}{U_0^2}\,x\sin\omega t, \qquad x < aU_0 t$$
$$f(x,t) = p\cdot\frac{\sigma\omega}{U_0}\,t\sin\omega t, \qquad x > aU_0 t \qquad (3.17)$$

As was to be expected, the variations of the parameter are influenced by two phenomena: on the one hand by the growth of the boundary layer during the movement of the plate from rest; on the other by small fluctuations in velocity on the outer boundary. It follows from formula (3.17), that two regions are formed on the plate: one for $X < aU_0t$, in which the boundary layer, i.e. the parameter, grew, and consequently also δ^* is dependent on x; the other when $x > aU_0t$, where the influence of the front edge is not yet operative, and where δ^* varies only with time. The dimensions of the second region progressively diminish, giving place to the first, thus letting the developing boundary layer gradually occupy the whole plate. The small velocity fluctuations, in their turn, bring about a break on the plate at some instant of F
time. If we use the condition (2.8) and the coefficients which correspond to 2
Hartree's profile (the broken line in Fig. 1), then we can conclude that the 2
break will begin in the second region, when $x > aU_0t$, at the instant of time t_0, determined by equation

$$\bar{t}_0 \sin N_{sh}\bar{t}_0 = -\frac{U_0}{\sigma}\frac{1}{N_{sh}} \qquad \left(\bar{t}_0 = \frac{U_0}{L}t_0\right) \tag{3.18}$$

where L is the length and $N_{sh} = \frac{\omega L}{U_0}$ Struhal's number.

Using (3.17) and the expression (2.7), we can determine the frictional stress and consequently the force of resistance F of a plate of length L, washed by the flow of fluid from two sides. Thus if we denote by s the wet surface of the plate and introduce the coefficient of resistance c_f according to the formula $F = \frac{1}{2} c_f s \rho U_0^2$, we shall get for C_f the following expression:

$$c_f = \frac{\sqrt{\bar{t}}}{\sqrt{N_{Re}}}\left[0.378 + \frac{1.06}{\bar{t}} + \frac{\sigma}{U_0} N_{sh}(1.06-0.125\,\bar{t})\,\sin N_{sh}\bar{t}\right] \tag{3.19}$$

where $N_{Re} = \frac{U_0 L}{\nu}$ is Reynold's number. At the instant $\bar{t} = 2.81$ the whole plate of length L will be occupied by a region in which the flow is dependent on x, and from this moment on, the formula for c_f will assume the form

$$c_f = \frac{1}{\sqrt{N_{Re}}}\left(1.27 + 1.19\frac{\sigma}{U_0} N_{sh} \sin N_{sh}\bar{t}\right) \tag{3.20}$$

In the case $\sigma = 0$, the formula (3.19) corresponds to the changes in the boundary layer on the plate, which started to move through impact with the constant velocity U_0. Together with this, at the instant $\bar{t} = 2.81$, when on the entire plate stationary movement is established, (3.19) is transformed into the expression (3.20), which in its turn almost coincides with the usual formula of Blasins /5/. Analyzing the relations we have obtained, we can come to the conclusion, that for $\sigma = 0$ the minimum value of the coefficient of resistance is reached, when a stationary state is established on the whole plate. We determine the magnitude A, which is equal to the momentum of the force of resistance F for the time $\bar{t} = 2.81$. Then considering continual motion, we will have

$$A = 4.75 \times \sqrt{\mu \rho L^3 U_0}.$$

On the other hand for damped movement we shall have

$$A = 3.58 \times \sqrt{\mu\rho L^3 U_0}.$$

It follows that the magnitude of the momentum of the force F during continual movement is 32.7% greater than the corresponding magnitude during stationary movement. Such an increase of the resistance force is due to the intensive whirl formation which takes place at the initial period of continual motion.

4. A Boundary Layer, Symmetric to the Axes. The computational method demonstrated above extends to the case of a streamlined body under revolving flow, directed parallel to the axis of symmetry of the body. As in the plane problem, in this case too it is possible to obtain a differential equation relative to $\varphi = \delta^{*2}/\nu$. Thus, assuming that the front part of the streamlined body is blunt, we shall have

$$\frac{\partial\varphi}{\partial t} + aU\frac{\partial\varphi}{\partial x} + \left(2a\frac{1}{R}\frac{dR}{dx}U + b\frac{\partial U}{\partial x} + c\frac{1}{U}\frac{\partial U}{\partial t}\right)\varphi = p \qquad (4.1)$$

where R(x) is the radius of any one of the parallels on the streamlined contour. Here, as above, the coefficients a, b, c, p depend on the family of velocity profiles that has been chosen.

As an example, we shall determine the beginning of a break in the rear critical point of a sphere of radius r, which at the instant t = 0 is set in motion on the boundary of the layer with the velocity $U = {}^3/_2 U_0 t^n \sin x/r$. In this case the break condition for the boundary layer will be written in the form

$$\left[\gamma\left(\frac{3}{2}\,\frac{U_0}{r}\,\frac{t^{n+1}}{n+1}\right)^{-1} - 1\right]\left\{\exp\left[\frac{3}{2}(2a+b)\,\frac{U_0}{r}\,\frac{t^{n+1}}{n+1}\right] - 1\right\} = -\frac{\alpha_2}{\alpha_3}\,\frac{2a+b}{p}, \qquad \gamma = \frac{n}{n+1} \qquad (4.2)$$

If in (4.2) we take α_1, equal to the break value h(f) and let n = 0, then a calculation based on the utilization of Hartree's function (the broken line) gives a break after the sphere passes through the distance $\frac{s_0}{r} = 0.31$. At the same time, with the aid of the relations given in Fig. 2 we get $\frac{s_0}{r} = 0.347$, and Targ's method /4/ yields $\frac{s_0}{r} = 0.31$. Comparing the given magnitude with the value $\frac{s_0}{2} = 0.39$, obtained by direct integration of the equations of the boundary layer /1/, we can conclude that in the given case the computation based on Fig. 2 and the right part of Table 1 is the most exact. Fig. 3 shows s_0/r against n for a sphere, as obtained according to formula (4.2), where for the original velocity distribution Hartree's function (the broken line) was taken. It follows from Fig. 3 that for any n the break on the sphere will occur later than on a circular cylinder of the same radius. This is explained by the fact that the negative pressure gradient at the rear critical point of a sphere is smaller than in the rear critical point of a cylinder. Just as in the plane case for the problems (3.9) and (3.11), the equation (4.1) can be reduced to an ordinary differential equation, if we take R(x) correspondingly in the form $R = \chi^n \lambda^n$ and $R = e^{\lambda n x}$.

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